# Math 304 (Spring 2015) - Homework 2

**Problem 1.**

Suppose $A = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 1 & -1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 4 & 1 \\ -2 & 3 & 1 \\ 2 & 0 & 4 \end{pmatrix}$. Compute

$$(a).\ A + 2B, \quad (b).\ AB, \quad (c).\ BA, \quad (d).\ (AB)^T.$$

**Solution:**

$$A + 2B = \begin{pmatrix} 1 & 11 & 6 \\ -4 & 8 & 7 \\ 5 & -1 & 10 \end{pmatrix}$$

$$AB = \begin{pmatrix} 2 & 13 & 20 \\ 6 & 6 & 22 \\ 6 & 1 & 8 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 7 & 22 \\ -1 & -1 & 9 \\ 6 & 2 & 16 \end{pmatrix}$$

**Problem 2.**

Given $A = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$, compute $A^2$, $A^3$, $A^4$ and $A^5$. Recall that for any natural number $k$,

$$A^k = \underbrace{AA \cdots A}_{k \text{ times}}$$

**Solution:**

$$A^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, A^3 = \begin{pmatrix} -\sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix}, A^4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix}$$

**Problem 3.**

Determine whether each of the following matrices has an inverse or not. If yes, find the inverse.

(a) $\begin{pmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & -6 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix}$

**Solution:**

(a) nonsingular. Its inverse is

$$\begin{pmatrix} -16 & -11 & 3 \\ 7/2 & 5/2 & -1/2 \\ -5/2 & -3/2 & 1/2 \end{pmatrix}$$

(b) singular

(c) nonsingular. Its inverse is

$$\begin{pmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{pmatrix}$$

**Problem 4.**

Solve the following linear system

$$\begin{cases} x_1 + x_3 = 1 \\ 3x_1 + 3x_2 + 4x_3 = 2 \\ 2x_1 + 2x_2 + 3x_3 = 1 \end{cases}$$

Could you use part (c) of the previous problem to solve the system?

**Solution:** The usual way to solve a linear system is to apply Gaussian elimination process (i.e. use elementary row operations to arrive at a row echelon form).

In this question, since we coefficient matrix is the same matrix as part (c) of the previous question, we can proceed as follows.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

We need to solve

$$A\mathbf{x} = \mathbf{b}$$

Since we know the inverse of $A$ from part (c) of the previous question, we have

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

Therefore, we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

**Problem 5.**

We know that if $A$ and $B$ are nonsingular, then $AB$ is also nonsingular. However, in general, the sum of two nonsingular matrices can be either nonsingular or singular.

(1) Find examples of $(2 \times 2)$ matrices $A$ and $B$ such that $A$ and $B$ are nonsingular, but $A + B$ is singular.

(2) Find examples of $(2 \times 2)$ matrices $A$ and $B$ such that $A$ and $B$ are nonsingular, and $A + B$ is also nonsingular, but $(A + B)^{-1} \neq A^{-1} + B^{-1}$.

**Solution:**

(1) for example,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

then

$$A + B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Both $A$ and $B$ are nonsingular, $A + B$ is singular.

(2) for example,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then

$$A + B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Notice that $A$, $B$, $A + B$ are nonsingular.

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (A + B)^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

So $A^{-1} + B^{-1} \neq (A + B)^{-1}$.